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On The Non-homogeneous Quinary Quintic Equation

$$x^4 + y^4 - (x+y)w^3 = 14z^2T^3$$

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Abstract:

The purpose of this paper is to examine the non-zero distinct integral solutions of quinary quintic non-homogeneous diophantine equation $x^4 + y^4 - (x+y)w^3 = 14z^2T^3$ in integers. In this paper, we present some different patterns of integral solutions to the above quintic diophantine equation in five variables through employing linear transformations.

Keywords: Quinary quintic equation , Non-homogeneous quintic equation, , Integer solutions

Introduction:

In the Number Theory and Mathematics, to find solutions of equations in integers is one of the oldest and significant mathematical problem since the second millennium B.C. ancient Babylonians who managed to find solutions of the equations systems with two unknowns. Different types of equations and systems were started to extend by Diophantus in third century A.D. Since then, many mathematicians have been working on the different types of Diophantine equations. Working on non-linear Diophantine equations of degrees higher than two worthy of notice success was acquired just in the 20th century.

It is well-known that the Diophantine equations ,homogeneous or non-homogeneous ,have aroused the interest of many mathematicians. In particular,one may refer [1-17] for quintic equations with three ,four and five unknowns .The above problems motivated us to search for the distinct integer solutions to quinary non-homogeneous quintic equation given by $x^4 + y^4 - (x+y)w^3 = 14z^2T^3$ and try to find the different sets of integer

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solutions to this diophantine equation by using elementary algebraic methods. The outstanding results in this study of diophantine equation will be useful for all readers.

Method of Analysis:

The quinary non-homogeneous quintic Diophantine equation to be solved for its distinct non-zero integral solutions is

$$x^4 + y^4 - (x + y)w^3 = 14z^2T^3. \qquad (1)$$

The substitution of the linear transformations

$$x = w + z\,,\ y = w - z, w \neq z \qquad (2)$$

in (1), leads to

$$z^2 + 6w^2 = 7T^3. \qquad (3)$$

which is solved through different approaches and different patterns of solutions thus obtained for (1) are illustrated below :

Pattern: 1

Assume

$$T = a^2 + 6b^2 \ ; a, b \neq 0 \ . \qquad (4)$$

Write the integer 7 on the R.H.S. of (3) as

$$7 = \left(1 + i\sqrt{6}\right)\left(1 - i\sqrt{6}\right) \ . \qquad (5)$$

Using (4), (5) in (3) and employing the method of factorization and equating positive factors we get,

$$\left(z + i\sqrt{6}w\right) = \left(1 + i\sqrt{6}\right)\left(a + i\sqrt{6}b\right)^3. \qquad (6)$$

Equating real and imaginary parts of (6), we get

$$\left.\begin{aligned} z &= f(a,b) - 6g(a,b)\,, \\ w &= f(a,b) + g(a,b)\,. \end{aligned}\right\} \qquad (7)$$

where

$$f(a,b) = a^3 - 18ab^2 \ , g(a,b) = 3a^2b - 6b^3$$

In view of (2), we obtain

$$\left.\begin{aligned} x &= 2f(a,b) - 5g(a,b)\,, \\ y &= 7g(a,b)\,. \end{aligned}\right\} \tag{8}$$

Thus (7), (8) and (4) represent non-zero distinct integer solutions to (1).

Note 1:

It is worth to observe that the integer 7 on the R.H.S. of (3) is also expressed as

The product of complex conjugates as below:

$$7 = \frac{(13+i\sqrt{6})(13-i\sqrt{6})}{25}\,,$$

$$7 = \frac{(29+i\sqrt{6})(29-i\sqrt{6})}{121}$$

Following the similar procedure as above , two more sets of integer solutions to (1) are obtained.

Pattern: 2

One may write (3) as

$$z^2 + 6w^2 = 7T^3 * 1. \tag{9}$$

Write the integer 1on the R.H.S. of (9) as

$$1 = \frac{\left(1+i2\sqrt{6}\right)\left(1-i2\sqrt{6}\right)}{25}\,. \tag{10}$$

Substituting (4), (5), (10) in (9) and employing the method of factorization, define

$$\left(z + i\sqrt{6}w\right) = \frac{1}{5}\left(1+i\sqrt{6}\right)\left(1+i2\sqrt{6}\right)\left(a+i\sqrt{6}b\right)^3\,. \tag{11}$$

Equating real and imaginary parts of (11), we get

$$\left.\begin{aligned} z &= z(a,b) = \frac{1}{5}\left(-11f(a,b) - 18g(a,b)\right)\,, \\ w &= w(a,b) = \frac{1}{5}\left(3f(a,b) - 11g(a,b)\right)\,. \end{aligned}\right\} \tag{12}$$

As our interest is on finding integer solutions, replacing a by 5A and b by 5B in (4) and (12), we get

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$$\left.\begin{aligned} z &= 5^2[-11f(A,B)-18g(A,B)], \\ w &= 5^2[3f(A,B)-11g(A,B)] , \\ T &= 25A^2+150B^2 . \end{aligned}\right\} \quad (13)$$

In view of (2), we obtain

$$\left.\begin{aligned} x &= 5^2[-8f(A,B)-29g(A,B)], \\ y &= 5^2[14f(A,B)+7g(A,B)] \end{aligned}\right\} \quad (14)$$

Thus (13) and (14) represent non-zero distinct integer solutions to (1).

Note 2:

Write the integer 1on the R.H.S. of (9) as

$$1=\frac{\left(5+i4\sqrt{6}\right)\left(5-i4\sqrt{6}\right)}{121},$$

$$1=\frac{\left(6r^2-s^2+i\sqrt{6}\,2rs\right)\left(6r^2-s^2-i\sqrt{6}\,2rs\right)}{(6r^2+s^2)^2}$$

$1=$

Following the similar procedure as above , two more sets of integer solutions to (1) are obtained.

Pattern: 3

Introduction of the transformations

$$T=6s, z=6sk \quad (15)$$

in (3) leads to

$$w^2=6s^2(42s-k^2) \quad (16)$$

Again, considering the transformations

$$s=42(6u^2+K^2), k=42K \quad (17)$$

in (15) and (16) , it is seen that

$$w=6*42^2u(6u^2+K^2), z=6*42^2K(6u^2+K^2), T=6*42\,(6u^2+K^2). \quad (18)$$

In view of (2) ,we have

$$x=6*42^2(u+K)(6u^2+K^2), y=6*42^2(u-K)(6u^2+K^2) \quad (19)$$

Thus, (18) and (19) give the integer solutions to (1).

Pattern :4

Introduction of the transformation

$$w = kT, k \geq 1 \tag{20}$$

in (3) leads to

$$z^2 = T^2(7T - 6k^2) \tag{21}$$

which is satisfied by

$$T = (7s^2 - 12s + 6)k^2 \ , z = (7s^2 - 12s + 6)k^3(7s - 6) \tag{22}$$

In view of (20) and (2) , one has

$$w = (7s^2 - 12s + 6)k^3, x = (7s^2 - 12s + 6)k^3(7s - 5), y = (7s^2 - 12s + 6)k^3(7 - 7s) \tag{23}$$

Thus ,(22) and (23) give the integer solutions to (1).

Note 3:

It is to be noted that (21) is also satisfied by

$$T = (7s^2 - 2s + 1)k^2 \ , z = (7s^2 - 2s + 1)k^3(7s - 1) \tag{24}$$

Correspondingly , we have

$$w = (7s^2 - 2s + 1)k^3, x = 7s(7s^2 - 2s + 1)k^3 \ , y = (7s^2 - 2s + 1)k^3(2 - 7s) \tag{25}$$

Thus ,(24) and (25) give the integer solutions to (1).

Remark :

It is worth mentioning that ,for a given value of k in (20) ,the corresponding two sets of integer solutions to (1) are obtained by substituting the considered value of k in (22) ,(23) and (24),(25). For example ,when k=2 ,we have the following two sets of integer solutions to (1) from (22),(23) and (24),(25):

Set 1: $$x = 8(7s^2 - 12s + 6)(7s - 5), y = 8(7s^2 - 12s + 6)(7 - 7s),$$
$$z = 8(7s^2 - 12s + 6)(7s - 6), w = 8(7s^2 - 12s + 6), T = 4(7s^2 - 12s + 6)$$

Set 2: $$x = 56s(7s^2 - 2s + 1) \ , y = 8(7s^2 - 2s + 1)(2 - 7s),$$
$$z = 8(7s^2 - 2s + 1)(7s - 1), w = 8(7s^2 - 2s + 1), T = 4(7s^2 - 2s + 1)$$

However,there are two more sets of integer solutions to (1) which are as given below:

Set 3: $x=(7s^2-10s+7)(7s-3), y=(7s^2-10s+7)(7-7s),$
$z=(7s^2-10s+7)(7s-5), w=2(7s^2-10s+7), T=(7s^2-10s+7)$

Set 4: $x=7s(7s^2-4s+4), y=(7s^2-4s+4)(4-7s),$
$z=(7s^2-4s+4)(7s-2), w=2(7s^2-4s+4), T=(7s^2-4s+4)$

Conclusion:

In this paper, we have made an attempt to find infinitely many non-zero distinct integer solutions to the non- homogeneous quintic equation with five unknowns given by $x^4+y^4-(x+y)w^3=14z^2T^3$. To conclude, one may search for other choices of solutions to the considered quintic equation with five unknowns and higher degree diophantine equations with multiple variables.

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